International Journal of Theoretical Physics, Vol. 33, No. 9, 1994

Magnetohydrodynamic Stabilization of a Stratified Cylindrical Flow

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Received September 5, 1993

The effect of a uniform axial magnetic field on the stability of a stratified cylindrical flow of negligible viscosity and resistivity is examined. The fluid is assumed to be electrically conducting. The basic density and velocity fields are allowed to vary in two directions. The standard normal mode approach is used to treat the stability of the system. The complex wave speed of an unstable mode lies in the upper half of a semicircle whose diameter decreases with increasing magnetic field. A strong enough magnetic field can completely stabilize flows with unstable density stratification.

1. INTRODUCTION

The problem of the stability of a circular fluid cylinder has attracted considerable interest both in plasma physics and in astrophysics. The classical work on the linear stability of a fluid column was performed by Lord Rayleigh (1878), who showed that a nonrotating column of inviscid fluid is unstable to axisymmetric disturbances whose wavelength in the axial direction is greater than the circumference of the column and stable to all nonaxisymmetric disturbances. Later Lord Rayleigh (1892) also considered the effect of viscosity and showed that in the limit of high viscosity the range of stable wave numbers is unaltered. For a nonrotating column, disturbances confined to planes perpendicular to the axis of the column are always stable. Recently, the effect of a uniform axial magnetic field on the capillary instability of a rigidly rotating fluid column has been investigated (Wilson, 1992).

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0020-7748/94/0900-1949$07.00/0

It is well known from liquid metal magnetohydrodynamics that for conducting liquid jets the magnetic field has the stabilizing effect of increasing the wavelength at which capillary instability appears and of decreasing the growth rate of unstable disturbances. In the theoretical case when the liquid is inviscid and of infinite electrical conductivity, it was found that the capillary instability can be completely suppressed by a magnetic field of sufficient strength (Chandrasekhar, 1961). The effect of a uniform axial magnetic field on the stability of cylindrical liquid bridges anchored between two conductor solid disks has been recently studied by Nicolás (1992).

The stability of stratified parallel flows has been extensively studied in view of its importance in oceanography and meteorology. A river flowing into the sea and a warm wind blowing over a cool one are two familiar examples of stratified flows which exist in our environment. Since the appearance of a paper by Howard (1961) on eigenvalue bounds for unstable waves in a plane—parallel flow of an inviscid incompressible stratified fluid, many subsequent papers generalizing the results have appeared in the literature. The extension of this problem to nondissipative parallel flows of an electrically conducting fluid permeated by an aligned magnetic field was made by Howard and Gupta (1962) for an incompressible fluid and Dandapat and Gupta (1975) for an incompressible fluid. Such studies of magnetohydrodynamic shear flows are relevant to the problem of the earth's magnetosphere.

However, in the above-mentioned studies the velocity profiles for the basic steady flows vary in one direction, i.e., there are single-coordinate-dependent. But in nature the velocity of parallel flows usually varies in more than one direction. Recently Gupta (1992) examined the linear stability of a parallel stratified flow of an incompressible, inviscid, perfectly conducting fluid in the presence of a uniform aligned magnetic field such that the basic density and velocity fields are allowed to vary in two directions. In the present work we extend the method used by Gupta (1992) for a rigid channel with rectangular cross section to include a rigid one with two concentric circles as cross section. This case is more interesting than the one studied by Gupta (1992).

2. PROBLEM FORMULATION AND LINEARIZED PERTURBATION EQUATIONS

We consider a layer of electrically conducting fluid confined between two vertical coaxial rigid cylinders which are infinitely long. Cylindrical polar coordinates (r, θ, z) are used, the z axis being taken upward along the common axis of the rigid cylinder. The inner and outer rigid cylinders are fixed. They are of radii R_1 and R_2, respectively. The layer is subjected to a uniform magnetic field of intensity H_0 directed along the common axis of the cylinders.

In order to study the hydromagnetic stability of the problem, the following additional assumptions will be considered:

(a) The basic flow is unidirectional with velocity $W_0(r, \theta)$ along the z axis and in the presence of a potential force field $G(r, \theta)$.
(b) The liquid is isothermal, inviscid, and incompressible, with density and pressure distributions $\rho_0(r, \theta)$ and $P_0(r, \theta)$, respectively, and its magnetic permeability is μ_e.
(c) The magnetic pressure is defined as $\Pi = P + \mu_e |H|^2/8\pi$, where P is the fluid pressure and H is the magnetic field.
(d) The displacement currents in Maxwell's equations are ignored.
(e) The linear stability theory is used by considering infinitesimal perturbations.

The pressure balance requirement gives

$$\frac{\partial \Pi_0}{\partial r} = \rho_0 \frac{\partial G}{\partial r}, \qquad \frac{\partial \Pi_0}{\partial \theta} = \rho_0 \frac{\partial G}{\partial \theta} \tag{1}$$

which lead to the following constraint on the density stratification along r and θ directions:

$$\frac{\partial \rho_0}{\partial \theta}\frac{\partial G}{\partial r} = \frac{\partial \rho_0}{\partial r}\frac{\partial G}{\partial \theta} \tag{2}$$

In the perturbed state the velocity, pressure, density, and magnetic field are taken as, respectively,

$$(\hat{u}, \hat{v}, \hat{w} + W_0), \qquad \Pi_0 + \hat{\Pi}, \qquad \rho_0 + \hat{\rho}, \qquad (\hat{H}_r, \hat{H}_\theta, \hat{H}_z + H_0)$$

Assuming the flow to be nondissipative, we have the linearized perturbation magnetohydrodynamic equations of momentum (Chandrasekhar, 1961)

$$\rho_0\left(\frac{\partial \hat{u}}{\partial t} + W_0 \frac{\partial \hat{u}}{\partial z}\right) = -\frac{\partial \hat{\Pi}}{\partial r} + \frac{\mu_e H_0}{4\pi}\frac{\partial \hat{H}_r}{\partial z} + \hat{\rho}\frac{\partial G}{\partial r} \tag{3}$$

$$\rho_0\left(\frac{\partial \hat{v}}{\partial t} + W_0 \frac{\partial \hat{v}}{\partial z}\right) = -\frac{1}{r}\frac{\partial \hat{\Pi}}{\partial \theta} + \frac{\mu_e H_0}{4\pi}\frac{\partial \hat{H}_r}{\partial z} + \frac{\hat{\rho}}{r}\frac{\partial G}{\partial \theta} \tag{4}$$

$$\rho_0\left(\frac{\partial \hat{w}}{\partial t} + \hat{u}\frac{\partial W_0}{\partial r} + \frac{\hat{v}}{r}\frac{\partial W_0}{\partial \theta} + W_0 \frac{\partial \hat{w}}{\partial z}\right) = -\frac{\partial \hat{\Pi}}{\partial z} + \frac{\mu_e H_0}{4\pi}\frac{\partial \hat{H}_z}{\partial z} \tag{5}$$

The equation of mass conservation is

$$\frac{\partial}{\partial r}(r\hat{u}) + \frac{\partial \hat{v}}{\partial \theta} + \frac{\partial}{\partial z}(r\hat{w}) = 0 \tag{6}$$

and the condition of incompressibility and nondiffusing impose

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{u}\frac{\partial \rho_0}{\partial r} + \frac{\hat{v}}{r}\frac{\partial \rho_0}{\partial \theta} + W_0 \frac{\partial \hat{\rho}}{\partial z} = 0 \tag{7}$$

The components of the magnetic induction equation are

$$\frac{\partial \hat{H}_r}{\partial t} + W_0 \frac{\partial \hat{H}_r}{\partial z} = H_0 \frac{\partial \hat{u}}{\partial z} \tag{8}$$

$$\frac{\partial \hat{H}_\theta}{\partial t} + W_0 \frac{\partial \hat{H}_\theta}{\partial z} = H_0 \frac{\partial \hat{v}}{\partial z} \tag{9}$$

$$\frac{\partial \hat{H}_z}{\partial t} + W_0 \frac{\partial \hat{H}_z}{\partial z} = \hat{H}_r \frac{\partial W_0}{\partial r} + \frac{\hat{H}_\theta}{r}\frac{\partial W_0}{\partial \theta} + H_0 \frac{\partial \hat{w}}{\partial z} \tag{10}$$

along with the solenoidal condition ($\nabla \cdot \mathbf{H} = 0$) for the magnetic field

$$\frac{\partial}{\partial r}(r\hat{H}_r) + \frac{\partial \hat{H}_\theta}{\partial \theta} + \frac{\partial}{\partial z}(r\hat{H}_z) = 0 \tag{11}$$

These are the equations which describe the linearized problem at hand. In spite of the enormous simplification which results from linearization, a general analysis of these equations is still lacking. They do, however, provide the starting point for a number of important investigations based on various additional simplifying assumptions.

A substantial simplification can be achieved by making a normal mode analysis in which all of the disturbances are assumed to depend on t and z through a factor of the form $\exp[ik(z - st)]$, where k is the wavenumber in the axial direction and the temporal exponent ($s = s_r + s_i$) will be complex and is to be determined in terms of the other parameters in the problem. The problem is linearly stable if $s_r \leq 0$ and linearly unstable if $s_r > 0$. Thus, we let

$$\hat{f}(r, z, \theta; t) = f(r, \theta) \exp[ik(z - st)] \tag{12}$$

where $\hat{f}$ stands for any perturbed quantity.

The amplitude of the normal modes satisfy the following equations:

$$\rho_0(ik(W_0 - s)u) = -\frac{\partial \Pi}{\partial r} + \frac{ik\mu_e}{4\pi} H_0 H_r + \rho \frac{\partial G}{\partial r} \tag{13}$$

$$\rho_0(ik(W_0 - s)v) = -\frac{1}{r}\frac{\partial \Pi}{\partial \theta} + \frac{ik\mu_e}{4\pi} H_0 H_\theta + \frac{\rho}{r}\frac{\partial G}{\partial \theta} \tag{14}$$

$$\rho_0\left(ik(W_0-s)w+u\frac{\partial W_0}{\partial r}+\frac{v}{r}\frac{\partial W_0}{\partial\theta}\right)=-ik\Pi+\frac{ik\mu_e}{4\pi}H_0H_z \tag{15}$$

$$\frac{\partial}{\partial r}(ru)+\frac{\partial v}{\partial\theta}+ikrw=0 \tag{16}$$

$$ik(W_0-s)\rho+u\frac{\partial\rho_0}{\partial r}+\frac{v}{r}\frac{\partial\rho_0}{\partial\theta}=0 \tag{17}$$

$$H_r=\frac{H_0u}{(W_0-s)} \tag{18}$$

$$H_\theta=\frac{H_0v}{(W_0-s)} \tag{19}$$

$$ik(W_0-s)H_z=H_r\frac{\partial W_0}{\partial r}+\frac{H_\theta}{r}\frac{\partial W_o}{\partial\theta}+ikH_0w \tag{20}$$

$$\frac{\partial}{\partial r}(rH_r)+\frac{\partial H_\theta}{\partial\theta}+ikrH_z=0 \tag{21}$$

It can be easily verified that when the expressions for H_r, H_θ, and H_z from (18)–(20) are substituted in (21), the resulting equation is consistent with (16). Further, the elimination of H_r, H_θ, and H_z from (15) and (18)–(20) yields

$$\rho_0\left[1-\frac{V_A^2}{(W-s)^2}\right]\left[ik(W_0-s)w+u\frac{\partial W_0}{\partial r}+\frac{v}{r}\frac{\partial W_0}{\partial\theta}\right]=-ik\Pi \tag{22}$$

where V_A, the Alfvén velocity in the basic state, is given by

$$V_A^2=\frac{\mu_eH_0^2}{4\pi\rho_0} \tag{23}$$

By making use of the substitutions

$$\eta=-\frac{iu}{k(W-s)},\qquad \zeta=-\frac{iv}{k(W-s)} \tag{24}$$

so we find that (17) becomes

$$\rho=-\left(\eta\frac{\partial\rho_0}{\partial r}+\frac{\zeta}{r}\frac{\partial\rho_0}{\partial\theta}\right) \tag{25}$$

Evidently η and ζ play the role of Lagrangian displacements. Substitution of (24) and (25) in (13) and (14) gives, on using (19) and (20),

$$\frac{\partial\Pi}{\partial r}=\left\{\rho_0k^2[(W_0-s)^2-V_A^2]-\frac{\partial\rho_0}{\partial r}\frac{\partial G}{\partial r}\right\}\eta-\frac{1}{r}\frac{\partial\rho_0}{\partial\theta}\frac{\partial G}{\partial r}\zeta \tag{26}$$

$$\frac{1}{r}\frac{\partial\Pi}{\partial\theta}=\left\{\rho_0k^2[(W_0-s)^2-V_A^2]-\frac{1}{r^2}\frac{\partial\rho_0}{\partial\theta}\frac{\partial G}{\partial\theta}\right\}\zeta-\frac{1}{r}\frac{\partial\rho_0}{\partial r}\frac{\partial G}{\partial\theta}\eta \tag{27}$$

Furthermore, when (16) and (24) are used in (22), the result is

$$\Pi = \rho_0[(W_0 - s)^2 - V_A^2]\left[\frac{1}{r}\frac{\partial}{\partial r}(\eta r) + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right] \tag{28}$$

The boundary conditions are the vanishing of the normal component of the velocity and the tangential component of the magnetic field on the rigid boundaries so that

$$\eta = 0 \qquad \text{and} \qquad \zeta = 0 \qquad \text{on} \quad r = R_1, R_2 \tag{29}$$

We now integrate the identity

$$\frac{1}{r}\left\{\frac{\partial}{\partial r}(r\bar{\eta}\Pi) + \frac{\partial}{\partial \theta}(\bar{\zeta}\Pi)\right\} = \left(\bar{\eta}\frac{\partial \Pi}{\partial r} + \frac{\bar{\zeta}}{r}\frac{\partial \Pi}{\partial \theta}\right) + \frac{\Pi}{r}\left(\frac{\partial}{\partial r}(r\bar{\eta}) + \frac{\partial \bar{\zeta}}{\partial \theta}\right) \tag{30}$$

over the flow domain D in the $x-y$ plane, where an overbar denotes the complex conjugate. By Green's theorem

$$\iint\limits_D \frac{1}{r}\left[\frac{\partial}{\partial r}(r\bar{\eta}\Pi) + \frac{\partial}{\partial \theta}(\bar{\zeta}\Pi)\right] da = \int_{r=R} \Pi(\bar{\eta} r\, d\theta - \zeta\, dr) + \int_{r=R} \Pi(\bar{\eta} r\, d\theta - \zeta\, dr) \tag{31}$$

where $da = r\, dr\, d\theta$ is the element area of the domain D. Clearly the line integral in (31) vanishes by virtue of the boundary conditions (29). Thus by (30)

$$\iint\limits_D \left(\bar{\eta}\frac{\partial \Pi}{\partial r} + \frac{\bar{\zeta}}{r}\frac{\partial \Pi}{\partial \theta}\right) da + \iint\limits_D \frac{\Pi}{r}\left(\frac{\partial}{\partial r}(r\bar{\eta}) + \frac{\partial \bar{\zeta}}{\partial \theta}\right) da = 0 \tag{32}$$

Substitution of (26)–(28) in (32) gives on simplification

$$\iint\limits_D [(W_0 - s)^2 - V_A^2] Q\, da = F \tag{33}$$

where

$$Q = \rho_0\left[k^2(|\eta|^2 + |\zeta|^2] + \frac{1}{r^2}\left|\frac{\partial}{\partial r}(r\eta) + \frac{\partial \zeta}{\partial \theta}\right|^2\right] \tag{34}$$

$$F = \iint\limits_D \frac{\partial G}{\partial \theta}\left(\frac{\partial \rho_0}{\partial \theta}\right)^{-1}\left|\eta\frac{\partial \rho_0}{\partial r} + \frac{\zeta}{r}\frac{\partial \rho_0}{\partial \theta}\right|^2 da \tag{35}$$

Clearly Q is positive definite.

3. INSTABILITY PARAMETERS

Consider the case of an unstable density stratification along the θ axis so that $\partial\rho_0/\partial\theta > 0$. Thus, if $\partial G/\partial\theta \geq 0$, then (35) shows that $F \geq 0$. Since (2) shows that $(\partial G/\partial r)(\partial\rho_0/\partial r)^{-1} = (\partial G/\partial\theta)(\partial\rho_0/\partial\theta)^{-1}$, it follows from (35) that when $\partial G/\partial r \geq 0$ and $\partial\rho_0/\partial r$ (or when $\partial G/\partial r \leq 0$ and $\partial\rho_0/\partial r < 0$) we must have $F \geq 0$. As a matter of fact, (2) implies that a stable or unstable stratification along the θ axis puts a constraint on the density stratification along the r axis.

Thus when $F \geq 0$, we find from (33) upon setting $s = s_r + s_i$ and equating real and imaginary parts

$$\iint_D [(W_0 - s_r)^2 - s_i^2 - V_A^2]Q\, da = F \geq 0 \tag{36}$$

$$2s_i \iint_D (W_0 - s_r)Q\, da = 0 \tag{37}$$

Let $(W_0)_{\max}$ and $(W_0)_{\min}$ be the upper bounds of $W_0(r, \theta)$ in D. Then clearly we obtain

$$\iint_D [W_0 - (W_0)_{\max}][W_0 - (W_0)_{\min}]Q\, da \leq 0 \tag{38}$$

Now using (37) in (36) gives

$$\iint_D (W_0)^2 Q\, da = F + \iint_D (S_r^2 + s_i^2 + V_A^2)Q\, da \tag{39}$$

Finally if we combine (38) and (39), we find

$$\{s_r^2 + s_i^2 + V_A^2 - s_r[(W_0)_{\max} + (W_0)_{\min}] + (W_0)_{\max}(W_0)_{\min}\} \iint_D Q\, da \leq -F \tag{40}$$

Since Q is positive definite and $F \geq 0$, it follows from (40) that

$$\{s_r^2 + s_i^2 + V_A^2 - s_r((W_0)_{\max} + (W_0)_{\min}) + (W_0)_{\max}(W_0)_{\min}\} \leq 0$$

which can be written as

$$\left\{s_r - \frac{1}{2}[(W_0)_{\max} + (W_0)_{\min}]\right\}^2 + s_i^2 \leq \left\{\frac{1}{2}[(W_0)_{\max} - (W_0)_{\min}]\right\}^2 - V_A^2 \tag{41}$$

Thus the complex wave speed s for the unstable wave (with $s_i > 0$) must lie in a semicircle in the upper half-plane with center at the point $(\frac{1}{2}[(W_0)_{\max} + (W_0)_{\min}], 0)$ and radius $(\{\frac{1}{2}[(W_0)_{\max} - (W_0)_{\min}]\}^2 - V_A^2)^{1/2}$.

Since the radius of this semicircle decreases with increase in V_A, it follows that an axial magnetic field exerts a stabilizing influence on the flow by reducing the zone of instability.

A novel fact emerging from (41) is that when $V_A > \frac{1}{2}[(W_0)_{\max} - (W_0)_{\min}]$ the semicircle (within which the complex wave speed of an unstable mode lies) collapses. Thus when the Alfvén speed of the axial magnetic field exceeds $\frac{1}{2}[(W_0)_{\max} - (W_0)_{\min}]$, the flow with a potentially unstable density stratification is completely stabilized.

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